The Mechanics and Graph-Theoretic Underpinnings of Simple Coloring in Sudoku

Sudoku Analytical Research

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Abstract

This report provides an exhaustive analysis of the **Simple Coloring** strategy (also known as Single's Chains). It explores the graph-theoretic foundations of the technique, specifically its reliance on bipartite matching and odd-cycle contradictions. By reducing the complex multi-value state of the grid into a binary system of "ON" and "OFF" possibilities, Simple Coloring serves as a formal proof system for verifying the bipartite consistency of candidate distributions.

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1 Introduction: Sudoku as a Constraint Environment

The game of Sudoku, while popularly conceived as a casual logic puzzle, represents a specific instance of the exact cover problem, a classical challenge in combinatorics and computer science. Formally, a standard 9×9 Sudoku grid imposes a set of strict constraints upon a graph of 81 vertices, requiring a proper 9-coloring such that no two vertices sharing an edge possess the same value.

As solvers progress beyond elementary scanning techniques—such as Hidden Singles or Naked Subsets—they encounter the need for global inference strategies. These strategies, collectively known as "chaining," rely on the propagation of logical implications across the grid. At the foundation of this hierarchy lies **Simple Coloring**. This technique serves as the bridge between local pattern recognition and complex graph traversal algorithms like Alternating Inference Chains (AICs). Simple Coloring is distinguished by its focus on a single digit's topology, reducing the grid state into a binary system of boolean possibilities.

2 Theoretical Foundations: The Algebra of Connectivity

To fully comprehend the mechanics of Simple Coloring, one must deconstruct the underlying logical structures. The efficacy of the strategy rests entirely on the properties of the links connecting candidate cells.

2.1 The Conjugate Pair (The Strong Link)

The primary operational unit in Simple Coloring is the **Conjugate Pair**. A conjugate pair exists for a specific digit k in a specific house H (row, column, or block) if and only if there are exactly two cells in H that contain k as a valid candidate.

Mathematically, let $\mathcal{C}_H(k)$ denote the set of cells in house H containing candidate k.

$$|\mathcal{C}_H(k)| = 2 \implies \text{Conjugate Pair}$$

Let the two cells be u and v. The relationship is biconditional:

1. **Forward Implication:** If u is the solution, then v cannot be. ($u \implies \neg v$)
2. **Reverse Implication (Strong Inference):** If u is *not* the solution, then v *must* be. ($\neg u \implies v$)

This second property defines a **Strong Link**, allowing the propagation of logic across the grid.

2.2 The Weak Link

In contrast, a **Weak Link** connects two candidates that share a house where $|\mathcal{C}_H(k)| > 2$.

- If x is True, then y must be False ($x \implies \neg y$).
- However, if x is False, we *cannot* infer that y is True.

Simple Coloring chains are built using Strong Links to ensure bidirectional validity, while eliminations often rely on the interaction of Weak Links with the chain.

2.3 Graph Topology of a Single Digit

When a solver initiates Simple Coloring for digit k, they extract a subgraph $G_k = (V, E)$:

- **Vertices (V):** All cells containing candidate k.
- **Edges (E):** Connections representing Conjugate Pairs.

The process of "coloring" is topologically equivalent to testing whether a connected component of this graph is **Bipartite**—divisible into two disjoint sets, $\mathcal{A}$ and $\mathcal{B}$, such that every edge connects a vertex in $\mathcal{A}$ to one in $\mathcal{B}$.

3 The Algorithmic Execution of Simple Coloring

3.1 Step 1: Identification and Seeding

The analysis begins with the selection of a "focus digit." Ideally, the digit appears "clumped" in pairs.

- **The Seed:** One cell of a Conjugate Pair is arbitrarily assigned **Color A** (e.g., Blue).
- This assignment is a hypothesis: "Let us assume this cell has parity A."

3.2 Step 2: Recursive Propagation

From the Seed, the solver follows the web of Strong Links:

1. **Propagation:** Any cell strongly linked to a Blue cell is assigned **Color B** (e.g., Green).
2. **Iteration:** Any cell strongly linked to a Green cell is assigned Blue.
3. **Recursion:** This continues until the chain terminates or no further strong links exist.

3.3 Logical States

Upon completion, the chain implies the universe must collapse into one of two states:

- **State α:** All **Blue** cells are True; all **Green** cells are False.
- **State β:** All **Green** cells are True; all **Blue** cells are False.

4 Elimination Rules: Logic and Proofs

4.1 Rule 2: The Color Wrap (Intra-Chain Contradiction)

Also known as "Twice in a Unit."

A Color Wrap occurs when two cells of the **SAME** color (e.g., two Blue cells) share a single house (row, column, or box).

Proof by Contradiction:

1. Assume the Blue color represents the True state.
2. This implies digit k appears twice in the same house.
3. This violates Sudoku rules.
4. Therefore, Blue cannot be True. It must be False.

Action: Eliminate candidate k from ALL Blue cells. By implication, ALL Green cells become the solution.

4.2 Rule 4: The Color Trap (Inter-Chain Elimination)

Also known as "Two Colors Elsewhere."

A Color Trap occurs when an uncolored candidate z shares a house with at least one **Blue** cell and at least one **Green** cell.

Logic Table:

Chain State	Blue Cell (u)	Green Cell (v)	Effect on Victim (z)
State A	**TRUE**	FALSE	Eliminated by Blue (u)
State B	FALSE	**TRUE**	Eliminated by Green (v)

Table 1: Logical Truth Table for Rule 4

Conclusion: In every valid solution, z is eliminated.

5 Relationships with Advanced Strategies

5.1 Simple Coloring vs. X-Cycles

Research indicates a structural isomorphism between Simple Coloring and **X-Cycles**.

- **Continuous X-Cycle:** A loop with an even number of nodes. Identical to a valid Simple Coloring chain where start and end connect.
- **Discontinuous X-Cycle:** A loop resulting in a contradiction (e.g., $x \implies \neg x$). Identical to the **Color Wrap** (Rule 2).

Simple Coloring is frequently described as "X-Cycles visualized," as the coloring process automatically reveals loops without requiring algebraic path tracking.

5.2 Comparative Taxonomy

Strategy	**Scope**	**Link Types**	**Relationship to SC**
Simple Coloring	Single Digit	Conjugate Pairs	Base Strategy
X-Cycles	Single Digit	Strong & Weak	Isomorphic (Loop view)
Turbot Fish	Single Digit	5 Strong Links	Subset of SC
3D Medusa	Multi-Digit	Conjugate & Bi-Value	Superset of SC

Table 2: Taxonomy of Chaining Strategies

6 Graph Theory Deep Dive

6.1 Bipartite Matching

The coloring process attempts to map a connected component of subgraph G_k onto the set $\mathbb{Z}_2$.

- **Even Cycles:** A chain that loops back consistently represents a stable, bipartite structure.
- **Odd Cycles:** A **Color Wrap** represents an odd cycle. If two Blue nodes share a house, they have a "weak" edge between them. The strong links provide a path of even length; the weak edge adds 1, creating an odd cycle. Bipartite graphs cannot contain odd cycles, proving the configuration is invalid.

7 Conclusion

Simple Coloring represents a pivotal moment in a Sudoku solver's progression. It is the point where the solver moves from "finding" to "proving." By reducing the complexity of the grid to a bipartite graph of strong links, Simple Coloring allows for the visualization of deep logical contradictions. Whether used to spot a Color Wrap or a Color Trap, the technique provides a robust proof system for candidate removal and serves as the pedagogical foundation for mastering 3D Medusa and AICs.